Adelic Line Bundles and Bogomolov's Conjecture

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These are some cursory notes for my talk in the Berkeley number theory student seminar. Due to lack of time and space, details of proofs will have to be omitted, but I have tried to include reasonably comprehensive sketches as well as references to full proofs. All errors and pedantry are due to me.

1 Introduction

The goal of these notes is to discuss the ideas that go into the proof of the Bogomolov conjecture by Ullmo–Zhang ([Ull98], [Zha98a]), and try to sketch a few of the steps. Our main references are [Zha95a], [Zha95b], [Zha98b], and [YZ24, Appendix A].

Let's set up the situation. Let A be an abelian variety over $\mathbf{Q}$. Recall that if $\mathcal{L}$ is an ample symmetric line bundle on A, we may associate a canonical Néron-Tate height function $h : A(\overline{\mathbf{Q}}) \to \mathbf{R}_{\geq 0}$, which is quadratic and homogeneous of degree 2. The following is the generalized Bogomolov conjecture proved by Zhang:

Theorem 1.1 ([Zha98a])**.** *If $X \subseteq A$ is a non-torsion subvariety (meaning it is not the translate of an abelian subvariety by a torsion point), then there exists $\epsilon > 0$ such that*

$$\{x \in X(\overline{\mathbf{Q}}) : h(x) \leq \epsilon\}$$

is not Zariski-dense.

Let's note two corollaries of this theorem. First is the "standard" version of the Bogomolov conjecture, proved by Ullmo:

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Corollary 1.2 ([Ull98])**.** Let C be a smooth projective (geometrically integral) curve over a number field K, of genus $g \geq 2$. Let D be a divisor of degree 1 on C (e.g. a rational point) and j_D the corresponding embedding of C into $J = J(C)$. Then there exists $\epsilon > 0$ such that

$$\{x \in j_D(C(\overline{\mathbf{Q}})) : h(x) \leq \epsilon\}$$

is finite.

Indeed any curve that is also an abelian variety must be genus 1, and a Zariski-dense set of closed points on a curve is the same as an infinite such set.

Remark 1.3. Since $h(x) = 0$ if and only if x is torsion [BG06, Theorem 9.2.10], this generalizes the Manin–Mumford conjecture.

2 Adelic Line Bundles

The idea of the overall approach is to interpret the Néron-Tate height in terms of suitable data of *metrized* line bundles, so as to obtain a better geometric understanding of a rather arithmetic construction. Recall that in the "classical" version of Arakelov theory, line bundles on arithmetic varieties (by which I mean a projective variety over $\operatorname{Spec}(\mathcal{O}_K)$ with other nice properties) are "compactified" by adding in archimedean data; that is, data over the "missing points" of $\operatorname{Spec}(\mathcal{O}_K)$ in a sense of the correspondence between prime ideals of $\mathcal{O}_K$ and places of K. Zhang's idea, introduced in [Zha93], is to also equip p-adic metrics at the *nonarchimedean* places of $\mathcal{O}_K$ as well. We will follow the construction in [Zha95b].

We begin in the local case. Let K be a local field that is either $\mathbf{C}$ or a p-adic field (the definitions below work for general complete valuation fields, but these are the cases we care about), and let X be a projective K-variety. In the case that $K = \mathbf{C}$, we have the usual notion of smooth (resp. continuous) metrized line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on X, which means that for each closed point $x \in X$, we equip $\mathcal{L}_x$ with a $\mathbf{C}$-norm that varies smoothly (resp. continuously) in x. This is the main object of classical Arakelov geometry.

We now turn to the nonarchimedean case, so let $\mathcal{O}_K$ be the valuation ring of K. Equip K with the normalized absolute value where $|\pi| = (N\pi)^{-1}$, for the product formula (to be used later on). This absolute value extends uniquely to $\overline{K}$. We write line bundles in additive notation, so $n\mathcal{L}$ means $\mathcal{L}^{\otimes n}$.

Definition 2.1. We say that $(\mathcal{X}, \mathcal{M})$ is a *projective model* of $(X, e\mathcal{L})$ if $\mathcal{X}$ is projective and flat over $\operatorname{Spec}(\mathcal{O}_K)$, the generic fiber of $\mathcal{X}$ is isomorphic to X, and $\mathcal{M}$ is a line bundle on $\mathcal{X}$ such that $(\mathcal{X}_K, \mathcal{M}_K) \cong (X, e\mathcal{L})$.

We now explain how to get a K-metric on $\mathcal{L}$ from this data, by which we mean a family of K-metrics on the fiber $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \overline{K}(x)$ over each geometric point $x \in X(\overline{K})$ that is Galois-invariant. Indeed, let $x \in X(\overline{K})$, so by taking the Zariski closure (or alternatively, valuative criterion of properness) we get a unique $\widetilde{x} \in \mathcal{X}(\mathcal{O}_{\overline{K}})$. Note that $\widetilde{x}^*\mathcal{M} \otimes_{\mathcal{O}_{\overline{K}}} K = x^*(e\mathcal{L})$, so that $\widetilde{x}^*\mathcal{M}$ is an $\mathcal{O}_{\overline{K}}$-lattice inside the (1-dimensional) $\overline{K}$-vector space $x^*(e\mathcal{L})$. Hence we get a norm on $x^*(eL)$ as follows:

$$\|l\|_{\mathcal{M}} \coloneqq \inf_{a \in \overline{K}} \{|a| : l \in a(\overline{x}^*\mathcal{M})\}.$$

Therefore we get a norm on $x^*\mathcal{L}$ via

$$\|l\|_{\mathcal{L}} \coloneqq \|l\|_{\mathcal{M}}^{1/e}.$$

In this way, we get a K-metric on $\mathcal{L}$.

Definition 2.2. The metric on $\mathcal{L}$ as constructed above is called a *model metric*, coming from the model $(\mathcal{X}, \mathcal{M})$.

Remark 2.3. Suppose $(\mathcal{X}', \mathcal{M}')$ is a model of $(X, \mathcal{L})$ dominating $(\mathcal{X}, \mathcal{M})$. Then it is not hard to see that the model metric on $\mathcal{L}$ induced from $(\mathcal{X}', \mathcal{M}')$ is the same as that of $(\mathcal{X}, \mathcal{M})$.

Definition 2.4. A metric on $\mathcal{L}$ is *continous* (resp. *bounded*) if there is a projective model $(\mathcal{X}, \mathcal{M})$ such that $\|\cdot\| / \|\cdot\|_{\mathcal{M}}$, as a well-defined function on $\overline{K}$-points, is continuous in the topology on $X(\overline{K})$ induced from the topology on $\overline{K}$ (resp. *bounded*).

We now turn to global fields. Let K be a number field, and let M_K be the set of places of K. Let X be a projective variety over K and let $\mathcal{L}$ be a line bundle on X.

Definition 2.5. An *adelic metric* on $\mathcal{L}$ is a collection of continuous bounded K_v-metrics $(\|\cdot\|_v)_v$ of $\mathcal{L}_{K_v}$ on X_{K_v}, for each $v \in M_K$, such that the following *coherence condition* is satisfied:

There is a nonempty open subset $U \subseteq \mathrm{Spec}(\mathcal{O}_K)$, a projective flat variety $\mathcal{X}$ on U with generic fiber X, a line bundle $\mathcal{M}$ on $\mathcal{X}$ extending $\mathcal{L}$, such that for all closed points $v \in U$, the K_v-metric $\|\cdot\|_v$ is induced by the model $(\mathcal{X} \times_U \mathcal{O}_{K_v}, \mathcal{M} \times_U \mathcal{O}_{K_v})$ as above.

We call $\overline{\mathcal{L}} \coloneqq (\mathcal{L}, (\|\cdot\|_v)_v)$ an *adelic line bundle* on X.

Example 2.6. Lets say we have a projective variety $\mathcal{X}$ over $\mathcal{O}_K$ with generic fiber X, and suppose we have a Hermitian line bundle $\mathcal{M}$ on $\mathcal{X}$ whose restriction to X is $e\mathcal{L}$ (by Hermitian line bundle we mean that there is a (smooth) Hermitian metric on $\mathcal{X}_{\mathbf{C}}$). Assume for convenience that $\mathcal{X}$ is also normal. Then $\mathcal{M}$ induces a metric $\|\cdot\|_p$ on $\mathcal{L}$ for all $p \nmid \infty$ in the same way as above, and also induces Hermitian metrics on $\mathcal{L}$. This clearly gives a collection of K_v-metrics on $\mathcal{L}$ which is also continuous and bounded.

Moreover, by spreading out a Cartier divisor corresponding to $\mathcal{L}$ (i.e. finding a common denominator for the finitely many equations and open sets), we find an nonempty open subset $V \subseteq \operatorname{Spec}(\mathcal{O}_K)$ such that $\mathcal{L}$ has an extension $\mathcal{L}_1$ on X_V. Then $e\mathcal{L}_1|_X = e\mathcal{L} = \mathcal{M}|_X$, so there is a nonempty open $U \subseteq V$ such that $e\mathcal{L}_1|_U = \mathcal{M}|_U$, and so for all $p \in U$, $\|\cdot\|_p$ is induced from $\mathcal{L}_1$. Hence the coherence condition is satisfied. The point is in particular is to check that there is an honest line bundle over the restriction of $\mathcal{X}$ to some nonempty open set, that restricts to $\mathcal{L}$ and induces the metrics on $\mathcal{L}$ (note $\mathcal{M}$ only restricts to a multiple of $\mathcal{L}$).

Such an adelic line bundle $\mathcal{L}$, constructed in this fashion, is called a *model adelic line bundle*. One thinks of model adelic line bundles as coming from a single model of (some multiple of) $\mathcal{L}$ defined over all of $\operatorname{Spec}(\mathcal{O}_K)$.

Definition 2.7. It will also be useful to take limits of adelic metrics. We say that a sequence $\|\cdot\|_n$ of adelic metrics *converges* to an adelic metric $\|\cdot\|$ (the *limit*) if there is an open subset U of $\operatorname{Spec}(\mathcal{O}_K)$ such that for each $p \in U$, $\|\cdot\|_{n,p} = \|\cdot\|_p$ for all n, and $\|\cdot\|_{n,p} / \|\cdot\|$ converges to 1 uniformly on $X(\overline{K}_p)$ for all p.

Definition 2.8. We say a model adelic line bundle induced by a model $(\mathcal{X}, \overline{\mathcal{M}})$ is *nef* if the Hermitian line bundle $\mathcal{M}$ is nef, in the sense that it has nonnegative degree on any curve contained in a special fiber, and the curvature form of $\mathcal{M}_{\mathbf{C}}$ on the complex manifold $X(\mathbf{C})$ is semipositive. An adelic line bundle on X is nef if it is isometric to the limit of a sequence of nef model adelic line bundles on X.

For the purposes of these notes we will call an adelic line bundle ample if its underlying line bundle is ample in the usual sense.

Finally, an adelic line bundle on X is *integral* if it is isometric to the difference of two nef adelic line bundles.

It might be a good idea to black-box these as some technical conditions needed to make the below theory work. I will try to point out where these conditions are needed without going into much detail, especially since there seem to be a lot of definitions, but it is not always clear where everything is used (at least to me). But for a "big–picture" view of things, see Remark 3.5.

Remark 2.9. The above constructions can be reinterpreted in a more "modern" way via Berkovich analytic spaces, which serve as the replacements for the complex manifolds at the finite places. We will not need this level of generality, but see the book [YZ24] for details.

3 Intersection Theory and Heights

We now define intersection theory of adelic line bundles and prove some useful results. For X a projective variety over a number field K, let $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_{d+1}$ be nef adelic line bundles on X. If $Z \subseteq X$ is a subvariety of dimension d, we will define an intersection product

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d+1} \cdot [Z] \in \mathbf{R}$$

as follows: assume firstly that each $\mathcal{L}_i$ is a model adelic line bundle. Hence $\mathcal{L}_i$ is obtained from a projective model $(\mathcal{X}_i, \overline{\mathcal{M}}_i)$ with $\mathcal{X}_i$ projective and flat over $\mathcal{O}_K$, $\mathcal{M}_i$ nef, and $\mathcal{M}_i|_X = e_i\mathcal{L}_i$. By taking the fiber product of all the $\mathcal{X}_i$'s and considering the Zariski closure of the image of X under the diagonal map into this product, we may assume all of the $\mathcal{X}_i$'s are the same (and then we take the corresponding pullbacks of the $\mathcal{M}_i$'s). In this case we define

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{d+1} \cdot [Z] \coloneqq \frac{1}{e_1 \cdots e_{d+1}} \overline{\mathcal{M}}_1 \cdots \overline{\mathcal{M}}_{d+1} \cdot [\overline{Z}]$$

with $\overline{Z}$ the Zariski closure of Z in $\mathcal{X}_1 = \ldots = \mathcal{X}_{d+1}$. Here the right-hand side is the usual intersection number on arithmetic varieties from Arakelov theory: inductively we have (see [YZ24, Section 2.1.5])

$$\overline{\mathcal{M}}_1 \cdots \overline{\mathcal{M}}_{d+1} \cdot [\overline{Z}] \coloneqq \sum_i a_i \overline{\mathcal{M}}_1 \cdots \overline{\mathcal{M}}_d \cdot [Z_i] - \int_{\overline{Z}(\mathbf{C})} \log \|s_{d+1}\| \, c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d),$$

where s_{d+1} is a nonzero rational section of $\mathcal{M}_{d+1}$ and $\sum_i a_i Z_i$ is the divisor of $s_{d+1}|_{\overline{Z}}$. The base case is given by the Arakelov theory for arithmetic surfaces (depending on whether Z is a vertical or horizontal divisor), as in [Mor14, Chapter 4].

In general, a nef adelic line bundle is the limit of a sequence of nef model adelic line bundles. So, to define the intersection product in general, we would like to take some sort of limit process. Here is the relevant result:

Theorem 3.1 (Zhang)**.** Let $\overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_{d+1}$ be nef adelic line bundles on X. Assume that $\|\cdot\|_i$, $1 \leq i \leq d+1$, is the limit of model adelic metrics induced by projective models $(\mathcal{X}_{i,n}, \overline{\mathcal{M}}_{i,n})$ with the $\mathcal{M}_{i,n}$ nef and $\mathcal{M}_{i,n}|_X = e_{i,n}\mathcal{L}_i$. Then

$$\overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_d \cdot [Z] \coloneqq \lim_{n_1, \ldots, n_d \to \infty} \frac{1}{e_{1,n_1} \cdots e_{d+1,n_{d+1}}} \overline{\mathcal{M}}_{1,n_1} \cdots \overline{\mathcal{M}}_{d+1,n_{d+1}} \cdot [Z]$$

exists and does not depend on the $(\mathcal{X}_{i,n}, \overline{\mathcal{M}}_{i,n})$. Therefore this defines a multilinear intersection product.

For the proof, see [Zha95b, Theorem 1.4]. Note that we can extend this intersection product via multilinearity to integrable line bundles.

Now we finally get to define heights. Let $\overline{\mathcal{L}}$ be a nef ample adelic line bundle and let $Z \subseteq X$ be a closed subvariety of dimension $d-1$. Then

Definition 3.2. With notation as above, the *height* of Z with respect to $\overline{\mathcal{L}}$ is given by

$$h_{\overline{\mathcal{L}}}(Z) \coloneqq \frac{(\overline{\mathcal{L}})^d \cdot [Z]}{d(\mathcal{L}^{d-1} \cdot [Z])} = \frac{\underbrace{\overline{\mathcal{L}} \cdots \overline{\mathcal{L}}}_{d \text{ times}} \cdot [Z]}{d(\underbrace{\mathcal{L} \cdots \mathcal{L}}_{d-1 \text{ times}} \cdot [Z])}.$$

Here the numerator is the adelic intersection product of d copies of $\overline{\mathcal{L}}$ against Z defined previously, and the denominator is the standard (classical) intersection product on varieties.

Example 3.3. As an example take $d = 1$, so that Z is a closed point $x \in X(\overline{K})$. Assume for convenience that $\overline{\mathcal{L}}$ is in fact a model adelic line bundle $(\mathcal{X}, \mathcal{M})$ with $\mathcal{M}$ nef and ample. Then the Zariski closure of x is some $\mathrm{Spec}(\mathcal{O}_F)$ for a finite extension $F = K(x)$ of K. Then $h_{\overline{\mathcal{L}}}(x)$ is equal to $\widehat{\deg}(\mathcal{M}|_{\overline{x}})/[F : K]$ since the denominator is just the degree of the point x. For a nonzero rational section s of $\mathcal{L}$ whose support does not contain x, it extends to a section $\overline{s}$ of $\mathcal{M}$, and we calculate

$$\begin{aligned}
\widehat{\deg}(\mathcal{M}|_{\overline{x}}) &= \widehat{\deg}\left(\mathrm{div}(\overline{s}|_{\overline{x}}), -\sum_{\sigma \in F(\mathbf{C})} \log \|s(x)\| \, [\sigma]\right) \\
&= \sum_{\mathfrak{p} \in \mathrm{maxSpec}(\mathcal{O}_F)} \mathrm{val}_{\mathfrak{p}}(\overline{s}|_{\overline{x}}) \log(N\mathfrak{p}) - \sum_{\sigma \in F(\mathbf{C})} \log \|s(x)\|_\sigma \\
&= -\sum_{\mathfrak{p} \in \mathrm{maxSpec}(\mathcal{O}_F)} \log \|s(x)\|_{\mathfrak{p}} - \sum_{\sigma \in F(\mathbf{C})} \log \|s(x)\|_\sigma \\
&= -\sum_{v \in M_F} \log \|s(x)\|_v^{e_v}
\end{aligned}$$

where $e_v = 2$ if v is a complex place of F (with complex conjugate embeddings identified as the same place), and is 1 otherwise. Therefore up to sign,

$$h_{\overline{\mathcal{L}}}(x) = -\frac{1}{[F : K]} \sum_{v \in M_F} \log \|s(x)\|_v^{e_v}$$

is a Weil height for X associated to the line bundle $\mathcal{L}$, up to the factor of $-1/[K : \mathbf{Q}]$ (i.e. it is the naive height associated to $\overline{\mathcal{L}}$, up to this factor and a bounded function on $X(\overline{K})$ that depends on the choice of s).[1]

Remark 3.4. [Zha95b, Section 2] applies the notion of the height of a subvariety to *polarized dynamical systems*, i.e. a projective variety X over a number field K, equipped with a self-map $f : X \to X$ such that there exists an ample line bundle $\mathcal{L}$ on X with $f^*\mathcal{L} = d\mathcal{L}$ for some $d \geq 2$. An example is $X = \mathbf{P}^n$, f a polynomial map of degree d, $\mathcal{L} = \mathcal{O}(1)$. Another common example is X an abelian variety, f the multiplication by 2 map, and $\mathcal{L}$ any symmetric line bundle. This is another major application of the theory.

Remark 3.5. As a concrete example of the previous remark as applied to the definition of height given in Definition 3.2, suppose that X is an abelian variety and $\mathcal{L}$ is symmetric. Then

[1] I may have screwed up the normalizations, so that this computation doesn't match what is in other sources. But the general idea of how we recover a usual Weil height associated to $\mathcal{L}$ should still be clear.

our construction gives the Néron-Tate height associated to $\mathcal{L}$ in the following sense. Recall that the Néron-Tate height associated to the ample symmetric line bundle $\mathcal{L}$ is constructed by taking a Weil height associated to $\mathcal{L}$ and defining $h(x) := \lim_{n\to\infty} 4^{-n} h_{\mathcal{L}}(2^n x)$. Since the Weil height machine only associates to a line bundle a height function *up to a bounded function*, it is not clear how to get at this canonical height function directly from $\mathcal{L}$. Zhang's theory allows us to do this with an adelic line bundle: roughly speaking, given an integral model $(\mathcal{X}, \overline{\mathcal{M}})$ of (X, L) (the Hermitian metrics on $\overline{\mathcal{M}}$ don't matter), the map $f = [2] : X \to X$ induces a map $f' : \mathcal{X} \to \mathcal{X}$ defined over a dense open, and we can pullback $\overline{\mathcal{M}}$ via repeated applications of these maps to get a sequence of line bundles $\overline{\mathcal{M}}_n$ on $\mathcal{X}$. The (model) adelic metrics induced by the $\overline{\mathcal{M}}_n$'s, as in Example 2.6, will then converge to an adelic metric on $\mathcal{L}$[2], and the height of this *adelic line bundle* gives the Néron-Tate height associated to $\mathcal{L}$, using the uniqueness of the Néron-Tate height as the only quadratic function in its equivalence class modulo bounded functions. This is proved in [Zha95b, Theorem 2.4].

The overall intuition is that the adelic line bundles are the correct setting in which these limiting arguments (e.g. for the intersection product to make sense) can be transported to geometric objects, while preserving the theorems for (classical) Hermitian line bundles (i.e. most of the theorems that are discussed below have classical counterparts). The coherence condition and the exact condition for the convergence of adelic metrics allow these arguments to continue working over places of bad reduction.

We now state the following theorem of Zhang. X is a projective variety of dimension $d-1$ over a number field K, and as usual $\overline{\mathcal{L}}$ is a nef and ample adelic line bundle on X.

Definition 3.6. Let $e_i(X)$, $1 \le i \le d$, be the *ith successive minimum* defined as

$$e_i(X) := \sup_{Y \subseteq X \text{ codim } i} \inf_{x \in (X-Y)(\overline{K})} h_{\overline{\mathcal{L}}}(x).$$

Here we say that the empty subvariety has codimension d.

As an example, when $d = 2$ (so X is a curve), e_1 is the essential minimum of heights of closed points on X (i.e. the infinimum of the height of all but finitely many points), and e_2 is simply the infimum of heights.

Theorem 3.7 (Theorem of successive minima, Theorem 1.10, [Zha95b])**.**

$$e_1(X) \ge h_{\overline{\mathcal{L}}}(X) \ge \frac{1}{d}\sum_{i=1}^{d} e_i(X).$$

[2]Note that the $\mathcal{M}_n$'s restrict to $4^n\mathcal{L}$ on X, and not to $\mathcal{L}$. This provides motivation for why we want to allow projective models to restrict to *multiples* of $\mathcal{L}$ on the generic fiber, and not insist that they restrict to exactly $\mathcal{L}$.

Notice that $h_{\overline{\mathcal{L}}}(X) = 0$ if and only if $e_1(X) = 0$. Therefore this theorem relates the Néron-Tate height of the whole variety X to the heights of its points. Ultimately this will be the idea used in the proof of the Bogomolov conjecture.

As an example, suppose X is a subvariety of an abelian variety A, which is equipped with a Néron-Tate height coming from an adelic line bundle $\overline{\mathcal{L}}$. If X is the translate of a proper abelian subvariety A' by a torsion point x, then the translates of the torsion points of $A'(\overline{K})$ remain torsion in X and are dense (in X), so $e_1(X) = 0$. Therefore $h_{\overline{\mathcal{L}}}(X) = 0$ by the theorem. On the other hand, if X is *not* the translate of an abelian subvariety by a torsion point, then from the definition we see that the (generalized) Bogomolov conjecture for X is equivalent to $e_1(X) > 0$ (since then we get an open subset of X containing points all of whose heights are at least $e_1(X)/2$). In turn this is equivalent to $h(X) > 0$, as observed above.

Example 3.8. In this example we follow [Zha95b, Section 3] and almost prove the classical Bogomolov conjecture. Use the notation of Theorem 1.2. In [Zha93], an "admissible dualizing sheaf" ω_a on C is defined, which is an adelic line bundle. The formal definition of "admissibility" is rather involved and requires the construction of certain measure on the metrized reduction graphs of (an integral model) of C. Unfortunately I do not have enough time to discuss this in detail during the talk. We will only state the following result: suppose that D_0 is a divisor of degree 1 such that $(2g-2)D_0$ is equivalent to the canonical divisor on C. Then

$$h_{\overline{\mathcal{L}}}(\phi(C)) = \frac{1}{8(g-1)}(\omega_a, \omega_a) + \left(1 - \frac{1}{g}\right) h_{\overline{\mathcal{L}}}(D - D_0).$$

It is known from [Zha93] that the admissible self-intersection number (ω_a, ω_a) is nonnegative. Therefore if $(2g-2)(D - D_0) = (2g-2)D - \omega_{C/K}$ is not torsion, then the above expression is strictly positive, which is the Bogomolov conjecture.

Proof of Theorem 1.1. Using Theorem 3.7 we prove the Bogomolov conjecture as in Theorem 1.1, following [Zha98a]. Let X be a subvariety of the abelian variety A, defined over a number field K, which is not a torsion subvariety. The proof first reduces to the case that $G(X) := \{a \in A : a + X = X\}$ is trivial—*this is where the fact that X is not torsion is used.*[3] In this case, we have the following lemma:

Lemma 3.9 (Lemma 3.1, [Zha98a])**.**

$$\alpha_m : X^m \to A^{m-1}, \quad \alpha_m(x_1, \ldots, x_m) = (x_1 - x_2, x_2 - x_3, \ldots, x_{m-1} - x_m)$$

[3] Sketch following [Mor14, pg. 268]: consider the quotient abelian variety $\pi : A \to A/G(X)$. Note that $G(\pi(X))$ is trivial by construction. So if the Bogomolov conjecture is true for $\pi(X) \subseteq \pi(A)$, and for every $\epsilon > 0$ the $\{x \in X(\overline{K}) : h(x) < \epsilon\}$ is dense in X, then the same is true on $\pi(X)$ (for any ample line bundle $\mathcal{L}'$ on $\pi(A)$ defining the Néron-Tate height we may find a large enough integer a such that $\pi^*(h_{L'}) \leq a h_L$). In that case $\pi(X)$ is a torsion subvariety, so X is also a torsion subvariety (a torsion point in $\pi(A)$ has a torsion point of A in its fiber). So we have reduced to the case of $\pi(X) \subseteq \pi(A)$ and $G(\pi(X)) = 0$.

is a generic embedding for m large enough, meaning that it is quasifinite with generic degree 1. In particular, it is birational onto its image.

Assume for contradiction that the Bogomolov conjecture is false for X. Therefore we may find a Zariski-dense sequence of points $x_1, x_2, \ldots$ that is *small*, i.e. whose heights converge to 0. Let $r : \mathbf{N} \to \mathbf{N}^m$ be a bijective map with components $r_1, \ldots, r_m$, and let $x(n) := (x_{r_1(n)}, \ldots, x_{r_m(n)})$ be a sequence of points on X^m. By construction this is also Zariski-dense on X^m. Therefore it contains a *generic* subsequence (meaning that no subsequence is contained in a proper subvariety of X^m), because as X is defined over a countable field, the set of proper subvarieties of X is countable. So we may simply select one $x(n_i)$ from the complement of each of these subvarieties to include in a subsequence, which is generic by construction.

By Lemma 3.9, we may assume that m large enough such that α_m is a generic embedding. Now let $\overline{\mathcal{L}}_A = (\mathcal{L}_A, \|\cdot\|)$ be an ample symmetric adelic line bundle on A that induces the given Néron-Tate height, whose Hermitian metric at some given archimedean place σ is *positive*[4]. We may define $\overline{\mathcal{L}}_X$ as the pullback of $\mathcal{L}_A$ to X, and $\overline{\mathcal{L}}_{X^m}$ (resp. $\overline{\mathcal{L}}_{A^{m-1}}$) as the product of the pullbacks of $\overline{\mathcal{L}}_X$ (resp. $\overline{\mathcal{L}}_A$) to X^m (resp. A^{m-1}) via the projection maps. Then with heights defined on X^m and A^{m-1} by these adelic line bundles, the $x(n_i)$ are a small sequence on X^m, and the $\alpha_m(x(n_i))$ are a small sequence on A^{m-1}. Therefore by Theorem 3.7, $h_{\overline{\mathcal{L}}_{X^m}}(X^m) = 0$ and $h_{\overline{\mathcal{L}}_{A^{m-1}}}(A^{m-1}) = 0$.

We now apply the following equidistribution theorem, which can be proved using the theorem of successive minima:

Lemma 3.10 (Theorem 2.1, [Zha98a])**.** Let X be a projective variety over a number field K equipped with a nef ample adelic line bundle $\overline{\mathcal{L}}$, let σ be a fixed embedding $K \hookrightarrow \mathbf{C}$, and assume that there is an embedding $X_\sigma(\mathbf{C}) \to Y$ into a complex projective variety Y with a strictly positive ample Hermitian line bundle $\overline{\mathcal{M}} = (\mathcal{M}, \|\cdot\|_0)$ that pulls back to $\overline{\mathcal{L}} = (\mathcal{L}_\sigma, \|\cdot\|_\sigma)$ on X_σ. Suppose $h_{\overline{\mathcal{L}}}(X) = 0$. Then for a generic and small sequence of points $\{x_n\}$ in $X(\overline{K})$, if $O(x_n)$ denotes the Galois orbit of x_n under the action of $\mathrm{Gal}(\overline{K}/K)$ (which we think about as the points lying above x_n in the base change $X_{\overline{K}}$), the $O(x_n)$ are equidistributed on the complex manifold $X_\sigma(\mathbf{C})$ with respect to the measure

$$dx := \frac{c_1(\mathcal{L}_\sigma, \|\cdot\|_\sigma)^{\dim(X)}}{\deg(\mathcal{L})}.$$

In other words, with

$$\delta_{x_n} := \frac{1}{|O(x_n)|} \sum_{y \in O(x_n)} \delta_y,$$

[4]To get $\overline{\mathcal{L}}_A$, we can start with an ample symmetric line bundle in the usual sense inducing the Néron-Tate height, equip it with a positive Hermitian metric [Mor14, Proposition 9.18], and perform the limiting process to produce an adelic line bundle as described in Remark 3.5.

the δ_{x_n} converge weakly to dx. This means that for any continuous function f on $X_\sigma(\mathbf{C})$,

$$\lim_{n\to\infty} \frac{1}{|O(x_n)|} \sum_{y\in O(x_n)} f(y) = \int_{X_\sigma(\mathbf{C})} f(x)dx.$$

Sketch. Let f be a continuous function on $X_\sigma(\mathbf{C})$. By the Stone-Weierstrass theorem, it suffices to prove the above limit in the case that f is the restriction of a smooth function g on Y. For positive λ, let $\|\cdot\|_\lambda$ denote the norm $\|\cdot\|_0 \exp(-\lambda g)$ on $\mathcal{M}$. Since $\|\cdot\|_0$ is assumed to be strictly positive, for all small enough λ, $\|\cdot\|_\lambda$ is also strictly positive by continuity. Let $\|\cdot\|'$ be the adelic metric on $\overline{\mathcal{L}}$ that is the same as the original adelic metric, except the metric at the place σ is $\|\cdot\|_\lambda$ instead of the original $\|\cdot\|_0$, and let $h'(Z)$ denote the height of a subvariety $Z \subseteq X$ with respect to this new adelic metric on $\mathcal{L}$. By the theorem of successive minima, we have

$$\liminf_{n\to\infty} h'(x_n) \geq e_1'(X) \geq h'(X)$$

since the x_n are generic (any proper subvariety of X only contains finitely many of the x_n).

Now unraveling the definition of height gives

$$h'(x_n) = h(x_n) + \lambda \int_{X_\sigma(\mathbf{C})} f\delta_{x_n}, \quad h'(X) = h(X) + \lambda \int_{X_\sigma(\mathbf{C})} f dx + O(\lambda^2).$$

Since the $h(x_n)$ tend to $h(X) = 0$, the inequality $\liminf_{n\to\infty} h'(x_n) \geq h'(X)$ gives

$$\liminf_{n\to\infty} \int_{X_\sigma(\mathbf{C})} f\delta_{x_n} \geq \int_{X_\sigma(\mathbf{C})} f dx.$$

If we replace f by $-f$ we get $\limsup_{n\to\infty} \int_{X_\sigma(\mathbf{C})} f\delta_{x_n} \geq \int_{X_\sigma(\mathbf{C})} f dx$, so that $\lim_{n\to\infty} \int_{X_\sigma(\mathbf{C})} f\delta_{x_n} = \int_{X_\sigma(\mathbf{C})} f dx$ as desired. □

Apply this lemma to the $x(n_i)$'s and the $\alpha_m(x(n_i))$'s (with Y being the complex points of a suitable power of A_σ). We obtain that the $\delta_{x(n_i)}$ converge to $dx_m \coloneqq p_1^*(dx)\cdots p_m^*(dx)$ with dx defined as above for $\overline{\mathcal{L}}_X$. Also, the $\delta_{\alpha_m(x(n_i))}$ converge to a corresponding measure dx_m' on $\alpha_m(X^m)$, and by the constructions we must have $dx_m = \alpha_m^*(dx_m')$, at least over a dense open set of X^m where α_m is birational. Then by continuity (in the analytic topology on $X^m(\mathbf{C})$), the two measures are the same on all of X.

Note that dx is a strictly positive form at any $x \in X$ by the Hermitian metric we originally put on $\mathcal{L}_A$. Then dx_m is strictly positive as well. On the other hand, α_m maps the diagonal of X^m is sent to $0 \in A^{m-1}$. This implies that $\alpha_m^*(dx_m')$ is not strictly positive at $(x, x, \ldots, x)$ (it is 0 along the diagonal), a contradiction. This completes the proof of the Bogomolov conjecture. □

Remark 3.11. In fact Zhang proves more in [Zha98a]: that with the setup in Lemma 3.10, the $O(x_n)$ actually converges to the normalized Haar measure on $A_\sigma(\mathbf{C})$.

References

[BG06] Enrico Bombieri and Walter Gubler, *Heights in diophantine geometry*, New Mathematical Monographs, Cambridge University Press, 2006.

[Mor14] Atsushi Moriwaki, *Arakelov geometry*, Translations of Mathematical Monographs, Volume 244, American Mathematical Society, 2014.

[Ull98] Emmanuel Ullmo, *Positivité et discrétion des points algébriques des courbes*, Annals of Mathematics **147** (1998), no. 1, 167–179.

[YZ24] Xinyi Yuan and Shou-Wu Zhang, *Adelic line bundles on quasi-projective varieties*, 2024.

[Zha93] Shou-Wu Zhang, *Admissible pairing on a curve.*, Inventiones mathematicae **112** (1993), no. 1, 171–194.

[Zha95a] ———, *Positive line bundles on arithmetic varieties*, Journal of the American Mathematical Society **8** (1995), no. 1, 187–221.

[Zha95b] ———, *Small points and adelic metrics*, J. Algebraic Geom. **4** (1995), 18.

[Zha98a] ———, *Equidistribution of small points on abelian varieties*, Annals of Mathematics **147** (1998), no. 1, 159–165.

[Zha98b] ———, *Small points and arakelov theory.*, Documenta Mathematica (1998), 217–225 (eng).